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# A Three Parameter Model for Birth Projections

A. ROMANIUK

The approach underlying the model proposed in this article constitutes a considerable departure from the conventional methods used for the projection of birth series. Instead of following the customary procedure of directly projecting the age-specific fertility rates, they can be derived with this model from only three relatively simple fertility measures, namely, total fertility rate, mean age of fertility, and modal age of fertility. The reduction of the number of fertility parameters offers appreciable operational and analytical advantages. Among these, the most significant is the fact that statistical manipulation is confined to only three fertility measures, all of which are particularly appropriate for the in-depth analysis which is required to provide a rationale for assumptions of future fertility.

The model is essentially an adaptation to projection requisites of the Pearson Type I Curve which, in light of the recent exploratory research on Canadian fertility data conducted by Mitra and Romaniuk,[1] appears to be particularly suitable for simulating age-specific fertility rates. The working out of a procedure which permits the parameters determining the relative age distribution of fertility to be derived from only two fertility measures – mean age and modal age – represented a decisive step forward in the construction of a parametric model based on the Pearson Type I Curve. The model presented in this paper can be viewed as a logical outcome of the exploratory work just mentioned. It is a fully operational model with particular adaptation for computer use. A wide collection of data on fertility and births for Canada is used to test the performance of the model, and rationales are presented for the selection of the three parameters utilized in making projections of fertility.

## FORMAL STRUCTURE OF THE MODEL

In order to calculate the number of births in any given year, one merely multiplies the number of women at each age by the fertility rate of the corresponding age, and then sums the products which are obtained. This technique can be represented symbolically in formula (1) in which the meaning of the different symbols is as follows:

$B$ = number of births,
$W$ = number of women,
$f$ = fertility rate,
$i$ = specification of age-group,
$t$ = time,
$\alpha$ = lower bound of reproductive period,
$\beta$ = upper bound of reproductive period.

$$B_t = \sum_{i=\alpha}^{\beta} W_{i,t} \cdot f_{i,t}. \tag{1}$$

In this formula the number of women, $W$, is provided independently by applying projected survival ratios to an initial population of women by age. Further discussion of this operation need not concern us here. The procedures which follow will be restricted to the second term of the

[1] S. Mitra and A. Romaniuk, 'Pearsonian Type I Curve and its Fertility Projection Potentials', Paper presented at the Annual Meeting of the American Statistical Association, Montreal, August 1972.

equation, namely, fertility rates, $f$. This term can be replaced by $F_t$ and $p_{i,t}$ which refer respectively to the total fertility rate and its proportional distribution by age. By substitution, formula (1) then becomes (2):

$$B_t = \sum_{i=\alpha}^{\beta} W_{i,t} \cdot F_t \cdot p_{i,t} \tag{2}$$

where

$$F_t = \sum_{i=\alpha}^{\beta} f_{i,t} \tag{3}$$

and

$$p_{i,t} = \frac{f_{i,t}}{\sum\limits_{i=\alpha}^{\beta} f_{i,t}}. \tag{4}$$

It has been demonstrated that adequate approximation of the fertility curve by age may be obtained by fitting a Type I Curve developed by K. Pearson[2]. Hence, one can generate the proportional distribution of fertility by age, $p_i$, by the application of the Type I Curve in the Pearsonian System; its functional form can be represented by

$$p_{i,t} = \frac{\left(1+\frac{x_i}{a_1}\right)^{m_1}\left(1-\frac{x_i}{a_2}\right)^{m_2}}{\sum\limits_{x_i=\alpha'}^{\beta'}\left(1+\frac{x_i}{a_1}\right)^{m_1}\left(1-\frac{x_i}{a_2}\right)^{m_2}} \tag{5}$$

where $\alpha' = \alpha - M$, and $\beta' = \beta - M$, $M$ being the modal age.

It may be noted that $x_i$ in the above equation refers to ages computed from the modal age of childbearing; consequently, the lower and upper bounds for $x_i$ would be $-a_1$ and $a_2$ respectively. The constants, $m_1$ and $m_2$ determine the shape of the curve. Moreover,

$$\frac{m_1}{a_1} = \frac{m_2}{a_2}. \tag{6}$$

In order to determine the relative distribution of fertility by age by means of the Pearson Type I Curve, values for the constants $a_1$, $a_2$, $m_1$ and $m_2$ are needed. These values may be calculated in different ways. For example, Elderton's method requires the first four moments of the distribution, average age, variance, index of asymmetry, and kurtosis.[3] By assuming that the period of procreation is fixed, for example, from 15 to 50, Mitra succeeded in deriving the four constants from only the first two moments of the distribution.[4] Recently, Mitra and Romaniuk have developed a procedure by which the constants may be derived from only the mean and modal age of fertility.[5] Because of its operational simplicity and analytical advantages, the latter procedure has been used in the construction of the model. These advantages will be discussed in greater detail in a later section of this article.

[2] S. Mitra, 'The pattern of age-specific fertility rates', *Demography*, 4, 2 (1967), pp. 894–906; K. Tekse, 'On demographic models of age-specific fertility rates', *Statistisk Tidskrift* (Stockholm), Third Series, 5, 3 (1967) pp. 189–207; N. Keyfitz, *Introduction to the Mathematics of Population* (Reading, Massachusetts: Adison-Wesley, 1968); R. C. Avery, 'Graduation of age-specific fertility rates', Third Conference on the Mathematics of Population Chicago, 1970 (mimeograph). Cf. also Mitra and Romaniuk, *loc. cit.* in previous footnote.

[3] W. P. Elderton, *Frequency Curves and Correlations* (Cambridge University Press, 1930).

[4] Cf. Mitra, *loc. cit.* in footnote 2.

[5] Cf. Mitra and Romaniuk, *loc. cit.* in footnote 1.

Let:

the average age of fertility be $A$;
The modal age of fertility be $M$;
the age at which women start childbearing be $\alpha$;
the age at which women terminate childbearing be $\beta$.

Then, the following relationships can be established:

$$\delta=\beta-\alpha \tag{7}$$

is the reproductive age interval, and

$$a_1=M-\alpha, \tag{8}$$

$$a_2=\delta-a_1=\delta-(M-\alpha), \tag{9}$$

$$\alpha'=\alpha-M=-a_1, \tag{10}$$

$$\begin{aligned}\beta'&=\beta-M=\delta+\alpha-M\\&=\delta-(M-\alpha)=a_2,\end{aligned} \tag{11}$$

$$m_2=\frac{a_2[(a_1+a_2)-2v]}{(a_1+a_2)(v-a_1)}, \tag{12}$$

where

$$v=A-\alpha, \tag{13}$$

$$m_1=\frac{a_1}{a_2}m_2. \tag{14}$$

Formulae (12) and (14) may then be replaced by (15) and (16) respectively:

$$m_2=\frac{[\delta-(M_t-\alpha)]\{\alpha-[2(A_t-\alpha)]\}}{\delta[(A_t-\alpha)-(M_t-\alpha)]}, \tag{15}$$

$$m_1=\frac{(M_t-\alpha)\{\delta-[2(A_t-\alpha)]\}}{\delta[(A_t-\alpha)-(M_t-\alpha)]}. \tag{16}$$

By substitution, formula (2) can now be rewritten as (17):

$$B_t=\sum_{i=\alpha}^{\beta}\left\{W_{i,t}\,.\,F_t\,.\left[\frac{\left(1+\frac{x}{a_1}\right)^{m_1}\left(1-\frac{x_i}{a_2}\right)^{m_2}}{\sum\limits_{x_i=-a_1}^{a_2}\left(1+\frac{x_i}{a_1}\right)^{m_1}\left(1-\frac{x_i}{a_2}\right)^{m_2}}\right]\right\}. \tag{17}$$

$$B_t=\sum_{i=\alpha}^{\beta}W_{i,t}\,.\,F_t\,.$$

$$\frac{\left(1+\frac{x_i}{M_t-\alpha}\right)^{\left(\frac{(M_t-\alpha)\{\delta-[2\,.\,(A_t-\alpha)]\}}{\delta[(A_t-\alpha)-(M_t-\alpha)]}\right)}\left(1-\frac{x_i}{[\delta-(M_t-\alpha)}\right)^{\left(\frac{[\delta-(M_t-\alpha)]\{\delta-[2(A_t-\alpha)]\}}{\delta[(A_t-\alpha)-(M_t-\alpha)]}\right)}}{\sum\limits_{x_i=-\alpha'}^{\beta'}\left(1+\frac{x_i}{M_t-\alpha}\right)^{\left(\frac{(M_t-\alpha)\{\delta-[2\,.\,(A_t-\alpha)]\}}{\delta[(A_t-\alpha)-(M_t-\alpha)]}\right)}\left(1-\frac{x_i}{[\delta-(M_t-\alpha)]}\right)^{\left(\frac{[\delta-(M_t-\alpha)]\{\delta-[2(A_t-\alpha)]\}}{\delta[(A_t-\alpha)-(M_t-\alpha)]}\right)}}. \tag{18}$$

Finally, by replacing the constants $a_1$, $a_2$, $m_1$ and $m_2$ with their determinants, formula (18), which is the expression for the model in the format that is suitable for computer use, may be obtained.

With $W_{i,t}$ given, and the limits of the reproductive age interval, that is, values of $\alpha$ and $\delta$, assumed to be fixed, the results of the model for a given year may then be obtained from only three *independent* fertility parameters: (1) total fertility rate, $F_t$; (2) average age of fertility, $A_t$; and (3) modal age of fertility, $M_t$.

However, the choice of values for $\alpha$ and $\delta$ is not arbitrary. Before making this choice, it is helpful to test against empirical data in order to discern which values for $\alpha$ and $\delta$ in the model produce the best results for a given population. It has been found, for example, that for Canada the best results are obtained when $\alpha = 17$ and $\delta = 33$. It should also be pointed out that the results of the model are more sensitive to the choice of the lower bound of the reproductive age interval, $\alpha$, than to the choice of the length of the procreative period, $\delta$.

## TESTS OF VALIDITY

The model has been submitted to a series of empirical tests using data for Canada. Thus, the annual number of births for the period 1926–1970, that is, for all the years for which the required data are available, were generated from the model and compared with the actual number of births for the same period. Similar operations have been performed for eight of the ten provinces; the two remaining provinces have been excluded because data are not available for long enough to make the test meaningful. Finally, the annual numbers of births derived from the model for Canada were compared with those obtained by the conventional procedure of directly projecting age-specific fertility rates. The three parameters required for the model – total fertility rate, average age of fertility, and modal age of fertility – were obtained from age-specific fertility rates, projected according to the assumptions specified below. The validity of the model will be judged empirically on the basis of the deviations of the derived births from the observed.

Although the data are limited to only one country, they nevertheless reflect demographic conditions of considerable diversity. During the course of the period examined, Canadian women experienced great changes in their fertility level and age pattern. For example, the total fertility rate varied from 2·6 in the mid-1930's to 3·9 in 1959, and by 1970 had dropped to 2·3. During the same period the average age of mothers at the time of the birth of their children gradually declined from 30 to 27 years. Provincial time series reflect an even greater variety of reproductive patterns. Throughout this period the age structure of women of childbearing age had also changed markedly.

As for prospective births, we have selected assumptions to represent both an unusually long period of up to 100 years in some cases, and a considerable diversity of demographic conditions. Thus, four fertility assumptions ranging from 1·9 to 2·8 births per woman, and an average age of maternity from 25·9 to 26·8 by 1985, have been selected. Migration is assumed to be zero, except for one assumption that there will be an annual net migration gain of 60,000 persons. Two mortality assumptions have been selected, one indicating that by the end of the century life expectancy at birth will have increased from its present level of 75 years to 85 years, and the other stating that there will be no change in mortality levels during this period.

The results of the tests are shown in the tables in the Appendix, and the reader should examine them for a more detailed analysis. It can be seen that, on the whole, the values derived from the model almost coincide with the actual values obtained from the conventional procedure. Among the several hundreds of deviations which were calculated there are only a few of 1%, and deviations of 2% are rare. Moreover, it is not certain that all the relatively larger deviations are due to internal defects of the model. In at least some of the cases, the deviations may have been caused by errors in the estimate of the modal age of fertility, which was approximated by means of a graphical adjustment of the values observed for the modal age.

For a more complete assessment of the model's performance, three observations should be noted.

First, closer scrutiny of the series of the calculated ratios of derived to actual number of births reveals variations in time that are cyclical in pattern. Thus, for several successive years the derived values are above and then for several years fall below the observed values. Presumably, this phenomenon is due to the effect of changes in the age distribution of women as a result of past variations in the birth rate.

Secondly, the smallness of the differences between the derived and actual total number of births is largely due to the fact that errors by age more or less cancel out. As the chart indicates, the ratios of derived to actual values vary in magnitude and sign for different age groups. Fortunately, because the deviations are large only for extreme ages of low fertility and are of opposite signs, their net effect is weak. Nevertheless, in order to evaluate the model's potential for projections, consideration must be given to the effect of the age pattern of biases inherent in it, in conjunction with the expected age distribution of females of reproductive age. The net difference between derived and actual births will increase if, for example, the number of females tends to increase considerably at those ages where the biases are particularly large.

Thirdly, to derive the annual number of births from the model, it is assumed that a woman's reproductive age begins at 17 and terminates at 50. Let us suppose that a different period, which more closely corresponds to current observations, such as ages 15 to 50, had been chosen. In order to evaluate the net effect of this latter choice, the following table compares the ratios of derived to actual number of births for a few selected years, under two different assumptions of the reproductive age span.

| | Period of procreation | |
|---|---|---|
| Year | 15–50 | 17–50 |
| 1926 | 1·005 | 1·000 |
| 1931 | 1·010 | 0·998 |
| 1941 | 1·000 | 0·996 |
| 1951 | 0·984 | 0·996 |
| 1961 | 1·013 | 1·002 |
| 1969 | 1·018 | 1·009 |

It appears that the choice of ages, 15 to 50, as the length of the reproductive period, results in a slight loss of accuracy, but by no means invalidates the model.

## RATIONALE BEHIND THE SELECTION OF THE PARAMETERS

The results of the tests presented in the above section lead to the conclusion that instead of following the conventional procedure of obtaining the annual number of births by directly projecting age-specific fertility rates, it is more advantageous to obtain them by using the model. Indeed, it is unnecessary and inefficient to examine and project each of the 35 age-specific fertility rates required to produce population projections by single years of age when virtually the same results can be obtained from only three parameters. This feature constitutes an important operational advantage of the model, but its merits are not confined to operational advantages alone. Another significant asset of this model is that it has inherent analytical properties, that is, assumptions about future fertility and its distribution are formulated in terms of the three parameters, and these parameters are amenable to in-depth analysis and meaningful demographic interpretation. An illustration is given in this Section of the manner in which plausible assumptions about the fertility parameters utilized in the model could be obtained.

Among the three parameters, the total fertility rate is particularly suitable for in-depth analysis. By making assumptions about future fertility in terms of the total fertility rate, it becomes possible to use cohort analysis to ascertain how changes in family size and the timing of births separately contribute to variations in the period total fertility rate. When attempting to project future fertility, it may be found advantageous to formulate assumptions about those cohorts of women who would bear children during the projection period in terms of their completed family size and the mean age at fertility. For the older cohorts who are still of childbearing age at the beginning of the projection period, information about their earlier fertility experience may be used to obtain their

remaining and expected completed fertility. This can be done either by simple graphical extrapolation or by some of the more elaborate mathematical procedures.[6] Further insight on changes in family size, and also on the age pattern of fertility as expressed by indices such as the mean or median age at childbearing, may be gained through analysis of parity – cohort data and the related parity progression ratios. Data on parity distributions in turn provide a suitable springboard for analysis in terms of the various social, economic and medical factors which affect procreation. Thus, the proportion of women who will have at least one child depends on factors such as the proportion of women who marry before reaching menopause, their age at marriage, the extent of their practice of contraception, and the incidence of sterility which in turn partially depends on the state of medical knowledge. Finally, although several authors have emphasized the difficulties of utilizing information on women's childbearing intentions as input to projections,[7] data from prospective surveys on childbearing intentions may be helpful in the formulation of assumptions about future family size.[8]

In following the process described above, the delicate task of translating assumptions about the family size and mean age of fertility for different cohorts of women into the period total fertility rate will become necessary. Unfortunately, the procedures which are generally used to transform cohort to period measures have not been very successful in projections. However, Ryder's type of cohort-to-period translation model[9] may be quite helpful for *approximating* the total fertility rate from the family size and mean age of fertility which particular cohorts of women are expected to experience in the future.

It should be stressed that the cohort approach advocated here should be viewed only as an analytical vehicle designed to help in formulating the specifications of assumptions which are stated in terms of period fertility measures. In order to generate annual numbers of births, period age-specific fertility rates are required. However, as has been argued above, the existing procedures for translating cohort measures into period age-specific fertility rates have proved to be inadequate. Consequently, no model is as yet operationally feasible, which is capable of projecting directly in terms of cohort measures. The experience with projections by the cohort fertility method used recently in population projections in the U.S.A. and Canada provides ample material to support this statement.[10]

The two remaining parameters, the mean and modal ages of fertility, may be dealt with more swiftly. In the light of empirical testing with Canadian data, it appears that the derived total number of births in a given year depends primarily on the total fertility rate and less on the distribution of the total fertility rate within the childbearing age span. That is, from the point of view of obtaining the correct total number of births, the errors in the mean and modal ages are less important than errors in the total fertility rate. Indications of future trends in the mean and modal

[6] N. Keyfitz, 'On future population', *Journal of the American Statistical Association*, **67**, 338 (1972), pp. 347–363; E. M. Murphy and D. N. Nagnur, 'A Gompertz fit that fits: Application to Canadian fertility patterns', *Demography* **9**, 1 (1972), pp. 35–50; A Romaniuk and S. Tanny, 'Projection of incomplete cohort fertility for Canada by means of the Gompertz Function', *Analytical and Technical Memorandum*, No. 1, Statistics Canada, Ottawa, 1969.

[7] N. B. Ryder and C. F. Westoff, 'The trend in the expected parity in the United States: 1955, 1960, 1965', *Population Index*, **33** (1967), pp. 153–168.

[8] L. Bumpass and C. F. Westoff, 'The prediction of completed fertility', *Demography*, **6**, 9 (1969), pp. 445–459; J. S. Siegel and D. S. Akers, 'Some aspects of the use of birth expectations data from sample surveys for population projections', *Demography*, **6**, 2 (1969), pp. 101–115.

[9] N. B. Ryder, 'The process of demographic translation', *Demography*, **1**, 1 (1964), pp. 74–82; R. Pressat, *L'analyse démographique*, 2<sup>e</sup> édition (Presses Universitaires de France, 1969).

[10] D. S. Akers, 'Cohort fertility versus parity progression as methods of projecting births', *Demography*, **2** (1965), pp. 414–428; L. O. Stone, 'Parametric Approaches of the Age Distribution of Cohort Total Fertility for Projections', Statistics Canada, Ottawa, July 1970 (mimeograph); A. Romaniuk, 'Fertility Projections by the Cohort Method for Canada 1969–84', *Analytical and Technical Memorandum*, No. 5, Census Division, Statistics Canada, November 1970.

ages can be obtained by examining within a period and/or cohort frame of reference such variables as age at marriage, parity distributions, and child-spacing patterns. An analysis of the time series of modal age usually indicates accidental fluctuations and age misreporting, and consequently it is advisable to remove these distortions by some smoothing operations before attempting to extrapolate future trends from this series. Such adjustments were required in the case of the data on modal age of fertility for Canada in the 1926–1970 period.

The shape of the curve which has been selected here to simulate the relative distribution of the age-specific fertility rates is determined by the nature of the relationship between the mean and modal ages of fertility.[11] Hence, particular care must be given to investigating the type of relationship which exists between these ages. Fortunately, for most countries with good registration records, there is usually a very high correlation between these two fertility measures. For Canada, a correlation coefficient of 0·98 has been obtained from the data for the 1926 to 1970 period, after adjustments were made for the irregular features in the modal age mentioned in the previous paragraph. In this case, the modal age for future years can be obtained from a regression model based on past data, provided that there is reason to believe the observed relationship between the two measures will continue to hold.

The preceding discussion outlining the manner in which one would proceed to obtain plausible assumptions about the fertility parameters used in the model should be viewed only as an illustration of the potentials for open-ended, in-depth analysis which is possible with it. In this approach, one begins with a set of simple and demographically meaningful indices – total fertility rate, modal age, and mean age – and then proceeds by stages to examine first the immediate and later the more remote variables which affect these indices. The ultimate goal of these endeavours would be an attempt to incorporate the three parameters upon which the model is based into a more complete socio-economic model.

In contrast to the model presented here, the conventional procedure of using age-specific fertility rates to project fertility offers little analytical possibility. Since an age-specific fertility rate has little meaning in itself, it is difficult to relate it to those factors which determine a woman's actual reproductive behaviour. Consequently, projections which are made directly in terms of period age-specific fertility rates are virtually restricted to making vague assumptions about future trends in fertility on the basis of historical trends in these rates.

In view of the fact that this model is based on simple and demographically significant parameters, it avoids, or at least minimizes, the difficulties of interpretation which are inherent in many of the mathematical models. These latter models are often ill-suited for demographic projections because they depend on parameters which are purely mathematical constructs devoid of demographic content.[12] In contrast to the purely mathematical models, the model presented in this paper does not have this limitation.

## CONCLUDING REMARKS

Any model which is intended to be truly valuable for projection purposes must meet two essential requirements. First, it must be suitable for simulating with sufficient accuracy the fertility experience of women in different age groups, and ultimately, the annual number of births. Secondly, it must have inherent explanatory properties – that is, for parametric models, it is imperative that the parameters used provide a demographic interpretation of fertility such that the

[11] I am grateful to R. Avery for stressing the significance of this aspect.

[12] J. P. Bongaards and W. D. O'Neil, 'A systems model for the population renewal process', *Demography*, 9, 2 (1972). Cf. also Stone, *loc. cit.* in footnote 10.

relationship between them and the fertility level and age pattern can be clearly understood. There might be differences in opinion as to the best approach to adopt in order to implement the latter requirement. Jan Hoem[13] has proposed interesting strategies, using mainly iterative techniques, such as Keyfitz's descent method,[14] in order to increase the graduation capabilities of a model without sacrificing too much of the provision for interpretation which is built into the model. A parametric model for estimating age-specific fertility rates in a simplified manner has been developed by Mazur.[15] Although his model appears to be particularly attractive for projection purposes because of its simplicity, it was not possible here to investigate more thoroughly its capability for satisfying the two essential requirements of a projection model specified above.

In the last two sections, an attempt was made to demonstrate that the model proposed in this paper satisfies these two fundamental criteria reasonably well. Thus, from only three fertility measures the annual number of births can be derived with extraordinary accuracy. Moreover, these parameters are easy to calculate and, it is argued, all have definite analytical advantages which facilitate the provision of meaningful demographic interpretations of fertility. These are used in turn to provide rationales for assumptions which are made about future fertility.

It is difficult to speculate on the degree of universality of this model's applicability because the tests of its validity were limited to data for only one country, namely, Canada. Before any definite statement can be made concerning the range of experiences to which the model may be applicable, it will be necessary to conduct tests similar to those illustrated in this paper with data from other nations. Moreover, additional testing is required to ascertain the model's sensitivity to the relationship between the mean and modal ages, for, as is argued elsewhere in this paper, this relationship is important for the shape of the fertility curve used here. Nevertheless, the conclusive results which were obtained from diverse demographic conditions in Canada suggest that use of this model is justified for the projection of births whenever there is evidence that the fertility age patterns conform to shapes represented by the system of Pearson Type I Curves.

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[13] J. M. Hoem, 'On the statistical theory of analytic graduation', *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability*, University of California, June and July 1970 (University of California Press), pp. 569–600.

[14] N. Keyfitz, *op. cit.* in footnote 2.

[15] P. D. Mazur, 'A demographic model for estimating age order specific fertility rates', *Journal of the American Statistical Association*, September 1963, pp. 774–788.

## APPENDIX

TABLE 1. *Model's input and output, Canada, 1926–1970*

| Year | Input*: $\alpha=17$, $\delta=33$ | | | Output | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Period total fertility rate $F$ | Mean age of (period) fertility $A$ | Mode age of (period) fertility $Mo$ | $a_1$ | $a_2$ | $m_1$ | $m_2$ | Derived number of births $B$ | Actual number of births $B'$ | Ratio of $B/B'$ |
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) |
| 1926 | 3·356 | 30·1 | 28·00 | 11·00 | 22·00 | 1·079 | 2·159 | 232,782 | 232,852 | 0·9997 |
| 1927 | 3·319 | 30·1 | 27·90 | 10·90 | 22·10 | 1·021 | 2·070 | 234,224 | 234,343 | 0·9995 |
| 1928 | 3·296 | 30·0 | 27·90 | 10·90 | 22·10 | 1·101 | 2·232 | 236,795 | 237,009 | 0·9991 |
| 1929 | 3·218 | 29·9 | 27·80 | 10·80 | 22·20 | 1·122 | 2·306 | 235,018 | 235,583 | 0·9976 |
| 1930 | 3·284 | 29·9 | 27·80 | 10·80 | 22·20 | 1·122 | 2·306 | 243,039 | 243,772 | 0·9970 |
| 1931 | 3·201 | 29·9 | 27·70 | 10·70 | 22·30 | 1·061 | 2·212 | 240,099 | 240,654 | 0·9977 |
| 1932 | 3·086 | 30·0 | 27·70 | 10·70 | 22·30 | 0·987 | 2·057 | 235,597 | 235,905 | 0·9987 |
| 1933 | 2·865 | 30·0 | 27·60 | 10·60 | 22·40 | 0·937 | 0·980 | 223·202 | 223,105 | 1·0004 |
| 1934 | 2·804 | 30·1 | 27·50 | 10·50 | 22·50 | 0·832 | 1·783 | 221,579 | 221,550 | 1·0001 |
| 1935 | 2·754 | 30·0 | 27·50 | 10·50 | 22·50 | 0·891 | 1·909 | 221,718 | 221,740 | 0·9999 |
| 1936 | 2·695 | 30·0 | 27·40 | 10·40 | 22·60 | 0·848 | 1·844 | 220,996 | 220,638 | 1·0016 |
| 1937 | 2·645 | 29·8 | 27·40 | 10·40 | 22·60 | 0·972 | 2·112 | 220,884 | 220,529 | 1·0016 |
| 1938 | 2·701 | 29·7 | 27·30 | 10·30 | 22·70 | 0·988 | 2·178 | 229,673 | 229,748 | 0·9997 |
| 1939 | 2·653 | 29·6 | 27·30 | 10·30 | 22·70 | 1·058 | 2·333 | 229,799 | 229,765 | 1·0002 |
| 1940 | 2·759 | 29·4 | 27·20 | 10·20 | 22·80 | 1·152 | 2·575 | 244,100 | 244,640 | 0·9978 |
| 1941 | 2·824 | 29·2 | 27·10 | 10·10 | 22·90 | 1·253 | 2·842 | 255,123 | 255,705 | 0·9977 |
| 1942 | 2·954 | 29·1 | 27·00 | 10·00 | 23·00 | 1·270 | 2·921 | 272,668 | 272,778 | 0·9996 |
| 1943 | 3·030 | 29·2 | 26·80 | 9·80 | 23·20 | 1·064 | 2·519 | 283,441 | 284,082 | 0·9977 |
| 1944 | 3·000 | 29·3 | 26·60 | 9·60 | 23·40 | 0·905 | 2·206 | 284,319 | 284,672 | 0·9988 |
| 1945 | 3·005 | 29·3 | 26·40 | 9·40 | 23·60 | 0·825 | 2·071 | 288,760 | 289,364 | 0·9979 |
| 1946 | 3·356 | 29·0 | 26·30 | 9·30 | 23·70 | 0·939 | 2·394 | 330,845 | 331,471 | 0·9981 |
| 1947 | 3·575 | 28·7 | 26·10 | 9·10 | 23·90 | 1·018 | 2·674 | 359,597 | 359,943 | 0·9990 |
| 1948 | 3·423 | 28·7 | 25·90 | 8·90 | 24·10 | 0·925 | 2·504 | 347,380 | 348,226 | 0·9976 |
| 1949 | 3·438 | 28·6 | 25·70 | 8·70 | 24·30 | 0·891 | 2·488 | 353,901 | 354,811 | 0·9974 |
| 1950 | 3·433 | 28·6 | 25·50 | 8·50 | 24·50 | 0·814 | 2·347 | 357,029 | 358,845 | 0·9949 |
| 1951 | 3·480 | 28·5 | 25·30 | 8·30 | 24·70 | 0·786 | 2·339 | 367,663 | 369,354 | 0·9954 |
| 1952 | 3·621 | 28·4 | 25·20 | 8·20 | 24·80 | 0·792 | 2·395 | 389,473 | 390,998 | 0·9961 |
| 1953 | 3·702 | 28·3 | 25·00 | 8·00 | 25·00 | 0·764 | 2·388 | 403,525 | 405,087 | 0·9961 |
| 1954 | 3·812 | 28·3 | 24·80 | 7·80 | 25·20 | 0·702 | 2·269 | 420,642 | 422,545 | 0·9955 |
| 1955 | 3·817 | 28·2 | 24·60 | 7·60 | 25·40 | 0·678 | 2·266 | 426,125 | 428,180 | 0·9952 |
| 1956 | 3·849 | 28·1 | 24·40 | 7·40 | 25·60 | 0·654 | 2·264 | 435,588 | 436,198 | 0·9986 |
| 1957 | 3·929 | 28·0 | 24·20 | 7·20 | 25·80 | 0·632 | 2·263 | 453,243 | 453,778 | 0·9988 |
| 1958 | 3·884 | 27·9 | 24·10 | 7·10 | 25·90 | 0·634 | 2·313 | 454,978 | 455,303 | 0·9993 |
| 1959 | 3·947 | 27·9 | 24·10 | 7·10 | 25·90 | 0·634 | 2·313 | 464,556 | 464,449 | 1·0002 |
| 1960 | 3·910 | 27·8 | 24·10 | 7·10 | 25·90 | 0·663 | 2·418 | 463,737 | 463,378 | 1·0008 |
| 1961 | 3·857 | 27·8 | 24·00 | 7·00 | 26·00 | 0·636 | 2·364 | 460,902 | 460,109 | 1·0017 |
| 1962 | 3·773 | 27·7 | 24·00 | 7·00 | 26·00 | 0·665 | 2·470 | 455,365 | 454,629 | 1·0016 |
| 1963 | 3·690 | 27·7 | 24·00 | 7·00 | 26·00 | 0·665 | 2·470 | 451,918 | 450,324 | 1·0035 |
| 1964 | 3·521 | 27·8 | 23·90 | 6·90 | 26·10 | 0·611 | 2·312 | 440,809 | 438,235 | 1·0059 |
| 1965 | 3·163 | 27·8 | 23·90 | 6·90 | 26·10 | 0·611 | 2·312 | 405,954 | 403,855 | 1·0052 |
| 1966 | 2·826 | 27·6 | 23·90 | 6·90 | 26·10 | 0·667 | 2·522 | 374,910 | 373,626 | 1·0034 |
| 1967 | 2·593 | 27·4 | 23·80 | 6·80 | 26·20 | 0·698 | 2·691 | 358,609 | 358,050 | 1·0016 |
| 1968 | 2·445 | 27·3 | 23·80 | 6·80 | 26·20 | 0·730 | 2·813 | 351,754 | 351,490 | 1·0008 |
| 1969 | 2·385 | 27·3 | 23·80 | 6·80 | 26·20 | 0·730 | 2·813 | 356,166 | 356,647 | 0·9987 |
| 1970 | 2·310 | 27·1 | 23·60 | 6·60 | 26·40 | 0·731 | 2·926 | 359,789 | 359,449 | 1·0010 |

* The figures regarding female population by age required for calculation of annual number of births have not been included in this table. They can be found in the Statistics Canada official publications.

TABLE 2. *Ratio of the derived number of births to the actual number of births by provinces, for Canada*

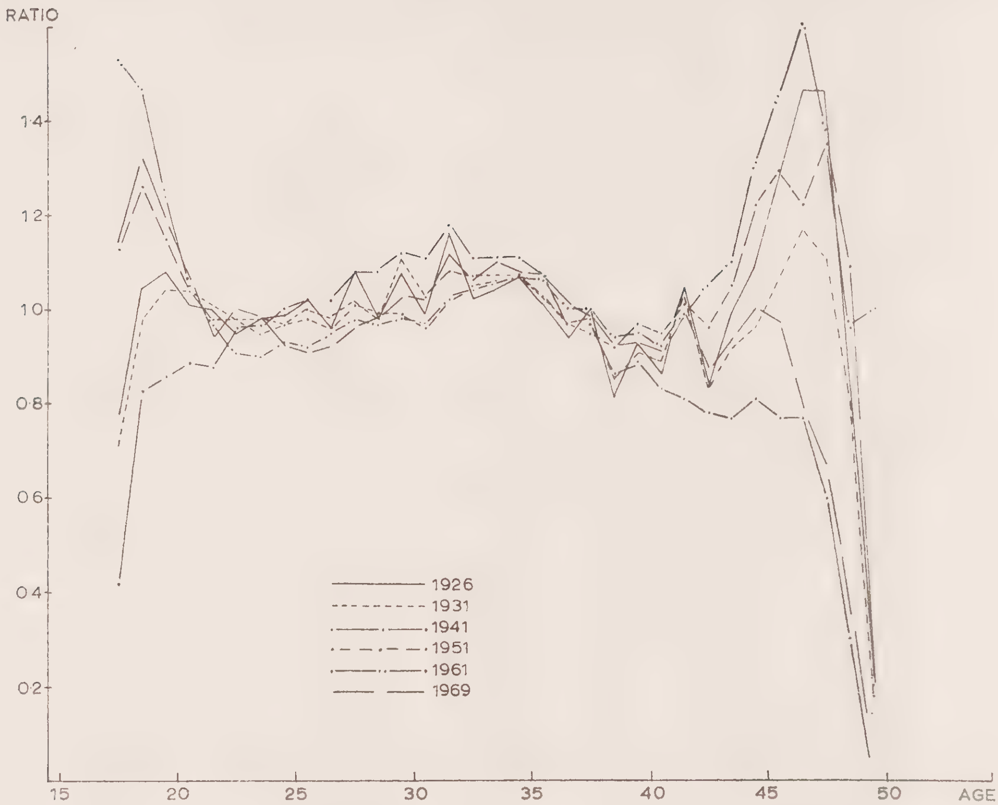
| Year* | N.S. | N.B. | Quebec | Ontario | Man. | Sask. | Alberta | B.C. |
|---|---|---|---|---|---|---|---|---|
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) |
| 1941 | 0·9989 | 0·99430 | 0·9986 | 0·9999 | 0·9966 | 0·9999 | 1·0032 | 1·0002 |
| 1949 | 0·9990 | 0·99660 | 0·9976 | 1·0016 | 0·9987 | 0·9906 | 1·0002 | 1·0066 |
| 1950 | 1·0078 | 0·99840 | 0·9968 | 0·9998 | 1·0017 | 0·9999 | 0·9999 | 1·0070 |
| 1951 | 0·9969 | 0·99980 | 0·9961 | 0·9923 | 0·9974 | 1·0014 | 0·9993 | 1·0075 |
| 1952 | 0·9985 | 0·99460 | 0·9917 | 0·9902 | 1·0057 | 1·0004 | 0·9980 | 1·0055 |
| 1953 | 0·9963 | 1·00060 | 0·9950 | 1·0000 | 1·0050 | 0·9992 | 0·9981 | 0·9961 |
| 1954 | 0·9995 | 0·99650 | 0·9960 | 1·0007 | 1·0027 | 0·9988 | 0·9982 | 0·9932 |
| 1955 | 1·0049 | 1·00300 | 0·9952 | 1·0004 | 0·9918 | 0·9993 | 0·9962 | 0·9927 |
| 1956 | 0·9961 | 1·00070 | 0·9930 | 0·9983 | 0·9984 | 1·0038 | 0·9951 | 0·9961 |
| 1957 | 0·9980 | 1·00710 | 0·9901 | 0·9967 | 1·0025 | 1·0010 | 1·0016 | 0·9921 |
| 1958 | 0·9967 | 1·00400 | 0·9966 | 0·9905 | 0·9909 | 0·9997 | 0·9952 | 0·9926 |
| 1959 | 0·9967 | 1·00370 | 0·9987 | 0·9957 | 0·9980 | 0·9999 | 1·0009 | 0·9927 |
| 1960 | 1·0002 | 1·00590 | 0·9992 | 0·9969 | 1·0004 | 1·0046 | 0·9977 | 1·0005 |
| 1961 | 1·0002 | 1·00640 | 1·0025 | 0·9988 | 1·0006 | 1·0035 | 0·9984 | 0·9999 |
| 1962 | 0·9969 | 1·00540 | 1·0050 | 1·0002 | 0·9990 | 1·0069 | 1·0000 | 1·0019 |
| 1963 | 0·9998 | 1·00440 | 1·0073 | 1·0021 | 0·9975 | 1·0058 | 1·0002 | 1·0002 |
| 1964 | 1·0021 | 0·99990 | 1·0104 | 1·0034 | 0·9984 | 1·0057 | 1·0019 | 0·9974 |
| 1965 | 0·9961 | 1·00470 | 1·0097 | 1·0019 | 1·0004 | 1·0013 | 0·9992 | 0·9927 |
| 1966 | 0·9850 | 1·00020 | 1·0076 | 0·9960 | 0·9933 | 0·9973 | 0·9961 | 0·9868 |
| 1967 | 0·9753 | 0·99780 | 1·0013 | 0·9969 | 0·9883 | 0·9822 | 0·9914 | 0·9873 |
| 1968 | 0·9743 | 0·99050 | 0·9971 | 0·9968 | 0·9836 | 0·9776 | 0·9900 | 0·9890 |
| 1969 | 0·9789 | 0·99393 | 0·9934 | 0·9983 | 0·9877 | 0·9800 | 0·9906 | 0·9901 |
| 1970 | 0·9842 | 0·98860 | 0·9954 | 1·0050 | 0·9833 | 0·9656 | 0·9836 | 0·9949 |

* Age-specific fertility rates required for the calculation of the model's parameters are not available for the 1942–1948 period.

TABLE 3. *Ratio of the number of births, derived by the model, to the number of births from conventional projections based on the following assumptions regarding the total fertility rate (F), the mean age (A), the modal age (M), the expectation of life ($e_0$) of female population, and the net migration (mig.) for Canada*

| Year | $F=1{\cdot}90$<br>$A=26{\cdot}80$<br>$M=24{\cdot}50$<br>$e_0=84{\cdot}50$<br>*mig.* = 0 | $F=2{\cdot}13$<br>$A=26{\cdot}80$<br>$M=24{\cdot}50$<br>$e_0=75{\cdot}70$<br>*mig.* = 60,000 | $F=2{\cdot}42$<br>$A=26{\cdot}60$<br>$M=22{\cdot}50$<br>$e_0=84{\cdot}50$<br>*mig.* = 0 | $F=2{\cdot}82$<br>$A=25{\cdot}90$<br>$M=22{\cdot}50$<br>$e_0=84{\cdot}50$<br>*mig.* = 0 |
|---|---|---|---|---|
| 1971 | 0·9907 | 0·9907 | 0·9910 | 0·9955 |
| 1972 | 0·9951 | 0·9952 | 0·9950 | 0·9992 |
| 1973 | 0·9962 | 0·9963 | 0·9960 | 0·9993 |
| 1974 | 0·9980 | 0·9983 | 0·9980 | 0·9995 |
| 1975 | 0·9999 | 1·0003 | 1·0003 | 0·9990 |
| 1976 | 1·0014 | 1·0021 | 1·0020 | 1·0020 |
| 1977 | 1·0019 | 1·0027 | 1·0030 | 1·0016 |
| 1978 | 1·0024 | 1·0036 | 1·0030 | 1·0020 |
| 1979 | 1·0027 | 1·0041 | 1·0040 | 1·0020 |
| 1980 | 1·0032 | 1·0048 | 1·0050 | 1·0023 |
| 1981 | 1·0043 | 1·0061 | 1·0060 | 1·0031 |
| 1982 | 1·0055 | 1·0074 | 1·0080 | 1.0033 |
| 1983 | 1·0068 | 1·0087 | 1·0100 | 1·0033 |
| 1984 | 1·0083 | 1·0102 | 1·0130 | 1·0128 |
| 1985 | 1·0094 | 1·0112 | 1·0150 | 1·0149 |
| 1986 | 1·0101 | 1·0118 | 1·0170 | 1·0159 |
| 1987 | 1·0106 | 1·0120 | 1·0180 | 1·0160 |
| 1988 | 1·0110 | 1·0120 | 1·0180 | 1·0153 |
| 1989 | 1·0111 | 1·0116 | 1·0160 | 1·0132 |
| 1990 | 1·0109 | 1·0108 | 1·0140 | 1·0106 |
| 1991 | 1·0103 | 1·0097 | 1·0100 | 1·0064 |
| 1992 | 1·0092 | 1·0081 | 1·0060 | 1·0021 |
| 1993 | 1·0073 | 1·0059 | 1·0010 | 0·9972 |
| 1994 | 1·0047 | 1·0032 | 0·9950 | 0·9924 |
| 1995 | 1·0017 | 1·0001 | 0·9900 | 0·9878 |
| 1996 | 0·9984 | 0·9970 | 0·9860 | 0·9838 |
| 1997 | 0·9956 | 0·9945 | 0·9830 | 0·9814 |
| 1998 | 0·9933 | 0·9925 | 0·9820 | 0·9801 |
| 1999 | 0·9916 | 0·9913 | 0·9820 | 0·9803 |
| 2000 | 0·9906 | 0·9911 | 0·9830 | 0·9818 |
| 2001 | 0·9903 | 0·9918 | 0·9850 | 0·9844 |
| 2011 | 1·0105 | | 1·0220 | |
| 2021 | 1·0029 | | 0·9920 | |
| 2031 | 0·9968 | | 1·0020 | |
| 2041 | 1·0098 | not available | 1·0130 | not available |
| 2051 | 0·9990 | | 0·9920 | |
| 2061 | 1·0024 | | 1·0100 | |
| 2071 | 1·0057 | | 1·0020 | |

See the text for further explanations of the underlying assumptions.




*Ratio of derived to observed number of births by age for few selected years, Canada*

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